**Signals & Systems**

**EEE-223**

Lab # 09



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| Name | Muhammad Haris Irfan |
| Registration Number | FA18-BCE-090 |
| Class | BCE-4A |
| Instructor’s Name | Muhammad Bilal Qasim |

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**LAB # 09**

**Complex Fourier Series Representation of Signals**

**Lab 09-** **Complex Fourier Series Representation of Signals**

**Pre Lab Tasks**

In this lab session, we will introduce a way of analyzing/decomposing a continuous time signal into frequency components given by sinusoidal signals. This process in crucial in the signal processing field since it reveals the frequency content of signal and simplifies the calculation of systems’ output. The analysis is based on the Fourier series. Up to this point, all signals were expressed in the time domain. With the use of Fourier series, a signal is expressed in the frequency domain and sometimes a frequency representation of a signal reveals more information about the signal than its time domain representation. There are three different and equal ways that can be used in order to express a signal into sum of simple oscillating functions, i.e., into a sum of sines, cosines, or complex exponentials. In this manual, symbols and  are often swapped in order for the code written in examples to be in accordance with the theoretical mathematical equations.

**9.1 Complex Exponential Fourier Series:**

Suppose that a signal  is defined in the time interval. Then,  is expressed in exponential Fourier series form (equation 9.1) as



where,

is the fundamental frequency, and is given by 

are real numbers

The terms  that appear in equation 9.1 are given by equation 9.2 as



The complex coefficients are called complex exponential Fourier series coefficients, while is a real number and is called a constant or dc component. Each coefficient corresponds to the projection of the signal at the frequency, which is known as harmonic. The Fourier series expansion is valid only in the interval, and the value of defines the fundamental frequency. As the Fourier series coefficients represent the signal in the frequency domain, they are also referred as the spectral coefficients of the signal.

**Example:**

Expand in complex exponential Fourier series the signal.

The first thing that need to be done is to define the quantities, and. Moreover, the signal is defined as symbolic expression.

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| Commands | Results | Comments |
| t0=0;  T=3;  w=2\*pi/T;  syms t  x=exp(-t);  ezplot(x,[t0 t0+T]), grid on | lab91.bmp | Definition and graph of signalin the time interval |

Afterwards, the coefficients are computed according to equation 9.2. Looking into equation 9.1, we observe that Fourier coefficients have to be calculated. Of course, this computation cannot be done in an analytical way. Fortunately, as the index approaches toward  or toward, the Fourier coefficients  are approaching zero. Thus,  can be satisfactorily approximated by using a finite number of complex exponential Fourier series terms. Consequently, by computing the coefficients  for , i.e., by using first 201 complex exponential terms, a good approximation of is expected. The approximate signal is denoted by , and is computed by equation 9.3 given as



|  |  |  |
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| Commands | | Comments |
| for k=-100:100  a(k+101)=(1/T)\*int(x\*exp(-\*k\*w\*t),t,t0,t0+T)  end | Calculation of coefficients according to equation 9.2. | |

In order to define the vector that contains the Fourier series coefficients, , the syntax is used for programming reasons, since in MATLAB the index of a vector cannot be zero or negative. Having calculated the coefficients, the signal is approximated according to equation 9.2, or more precisely according to equation 9.3. Note that equation 5.2 is sometimes called the synthesis equation, while we in this manual equation 9.3 is referred as analysis equation.

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| Commands | Results | Comments |
| for k=-100:100  ex(k+101)=exp(j\*k\*w\*t);  end  xx=sum(a.\*ex)  ezplot(xx,[t0 t0+T]), grid on  title('Approximation with 201 terms') | lab92.bmp | Initially the quantities are computed and afterward the signal is approximated according to equation 9.3. |

The plotted signal that is computed with the use of the complex exponential Fourier series is almost identical with the original signal . In order to understand the importance of number of terms used for approximation of original signal, the approximate signal is constructed for different values of . First, the signal is approximated by three exponential terms, i.e., the coefficients are computed for .

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| Commands | Results | Comments |
| clear a ex;  for k=-1:1  a(k+2)=(1/T)\*int(x\*exp(-j\*k\*w\*t),t,t0,t0+T);  end  for k=-1:1  ex(k+2)=exp(j\*k\*w\*t);  end  xx=sum(a.\*ex);  figure();  ezplot(xx,[t0 t0+T]), grid on  title('Approximation with 3 terms') | lab93.bmp | When 3 terms are used in approximation of by ,i.e., , the approximation signal is pretty dissimilar from the original signal . |

Next, the signal is approximated by 11 exponential terms, i.e., the coefficients are computed for .

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| --- | --- | --- |
| Commands | Results | Comments |
| for k=-5:5  a(k+6)=(1/T)\*int(x\*exp(-j\*k\*w\*t),t,t0,t0+T);  end  for k=-5:5  ex(k+6)=exp(j\*k\*w\*t);  end  xx=sum(a.\*ex);  figure();  ezplot(xx,[t0 t0+T]), grid on  title('Approximation with 11 terms') | lab94.bmp | It is clear that even when 11 terms are used, namely, , the approximation of by is not good and is pretty dissimilar to . |

Finally, the signal is approximated by 41 exponential terms, i.e., the coefficients are computed for .

|  |  |  |
| --- | --- | --- |
| Commands | Results | Comments |
| for k=-20:20  a(k+21)=(1/T)\*int(x\*exp(-j\*k\*w\*t),t,t0,t0+T);  end  for k=-20:20  ex(k+21)=exp(j\*k\*w\*t);  end  xx=sum(a.\*ex);  figure();  ezplot(xx,[t0 t0+T]), grid on  title('Approximation with 41 terms') | lab95.bmp | The signal is now computed from 41 terms and is starting to look quite similar to the original signal. Thus, this is a quite satisfactory approximation. |

From the above analysis, it is clear that when many exponential terms are being considered in the construction of the approximate signal, a better approximation if the original signal is obtained. As illustrated in the beginning of the example, when 201 terms were used for the construction of the approximate signal, the obtained approximation was very good.

**9.2 Plotting Fourier Series coefficients:**

In the previous section, the Fourier coefficients were computed. In this section, the way of plotting them is presented. Once again, the signal the signal is considered. The coefficients of the complex exponential form are the first that will be plotted for and for . In the usual case, the coefficients are complex numbers. A complex number  can be expressed as , where  is the magnitude and is the angle of . Therefore, in order to create the graph of the coefficients of the complex exponential form, the magnitude and the angle of each coefficient have to be plotted.

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| Commands | Results | Comments |
| syms t k n  x=exp(-t);  t0=0;  T=3;  w=2\*pi/T;  a=(1/T)\*int(x\*exp(-j\*k\*w\*t),t,t0,t0+T);  k1=-6:6;  ak=subs(a,k,k1);  stem(k1,abs(ak));  legend('|a\_k|, k=-6:6') | lab95.bmp | The signal is defined and the magnitude of the coefficients is plotted for . |
| stem(k1,angle(ak));  legend('\angle a\_k, k=-6:6') | lab95.bmp | The angles of coefficients are plotted for . It is worth noticing the way that the angle symbol is drawn through the legend command. |
| k2=-40:40;  ak2=subs(a,k,k2);  stem(k2,abs(ak2));  legend('|a\_k|, k=-40:40') | lab95.bmp |  |
| stem(k2,angle(ak2));  legend('\angle a\_k, k=-40:40') | lab95.bmp |  |

**9.3 Fourier Series of Complex Signals:**

So far, we were dealing with real signals. In this section, we examine the Fourier series representation of a complex valued signal.

**Example:**

Compute the coefficients of the complex exponential Fourier series and the trigonometric Fourier series of the complex signal. Moreover, plot the approximate signals using5 and 41 components of the complex exponential form.

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| Commands | Results/Commands |
| syms t  t0=0;  T=10;  w=2\*pi/T;  x=t^2+j\*2\*pi\*t;  subplot(211)  ezplot(real(x),[t0 T]),grid on;  title('Real part of x(t)');  subplot(212)  ezplot(imag(x),[t0 T]),grid on;  title('Imaginary part of x(t)'); | The signal  is complex; hence, its real and imaginary parts are plotted separately.  lab95.bmp |
| for k=-2:2  a(k+3)=(1/T)\*int(x\*exp(-j\*k\*w\*t),t,t0,t0+T);  ex(k+3)=exp(j\*k\*w\*t);  end  xx=sum(a.\*ex);  subplot(211)  ezplot(real(xx),[t0 T]),grid on;  title('Real part of xx(t)');  subplot(212)  ezplot(imag(xx),[t0 T]),grid on;  title('Imaginary part of xx(t)'); | Computation of the first five coefficients of complex exponential form and graph of the approximate signal is obtained with five terms. The coefficients of the complex exponential form are complex, thus their magnitude and phase are plotted.  lab96.bmp |
| a1=eval(a)  subplot(211);  stem(-20:20,abs(a1));  legend ('|a\_k| ,k=-20:20')  subplot(212);  stem(-20:20,angle(a1));  legend ('\angle a\_k ,k=-20:20') | lab97.bmp |

Evaluate the approximation by 41 components yourself.

**9.3 Fourier Series of Periodic Signals:**

In the previous sections, the Fourier series expansion of a signal was defined in a close time interval. Beyond this interval, the Fourier series expansion does not always converge to the original signal. In this section, we introduce the case where the signalis a periodic signal with period , i.e., . In this case, the Fourier series is also periodic with period ; thus converges to for .

**Example:**

Approximate by Fourier series, the periodic signal that in one period is given by



First the signal is plotted over the time of five periods for reference reasons.

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| Commands | Results/Comments |
| t1=0:.01:1;  t2=1.01:.01:2;  x1=ones(size(t1));  x2=-ones(size(t2));  x=[x1 x2];  xp=repmat(x,1,5);  t=linspace(0,10,length(xp));  plot(t,xp) | lab97.bmp |

Afterwards, the two part signal is defined as single symbolic expression, where is the unit step function. Note that the periodic signal is entirely determined by its values over one period. Thus, the symbolic expression of is only defined for time interval of interest, namely, (one period). The defined symbolic expression is plotted in one period for confirmation.

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| Commands | Results |
| syms t  x=heaviside(t)-2\*heaviside(t-1);  ezplot(x,[0 2]); | lab98.bmp |

Finally, the complex exponential Fourier series coefficients are calculated and the approximate signal is computed and plotted for , i.e., for time of five periods. As in previous examples, is computed and plotted for various number of exponential terms used.

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| Commands | Results |
| k=-2:2;  t0 =0;  T=2;  w=2\*pi/T;  a=(1/T)\*int(x\*exp(-j\*k\*w\*t),t,t0,t0+T)  xx=sum(a.\*exp(j\*k\*w\*t))  ezplot(xx,[0 10])  title('approximation with 5 terms') | lab99.bmp |
| k=-5:5;  a=(1/T)\*int(x\*exp(-j\*k\*w\*t),t,t0,t0+T);  xx=sum(a.\*exp(j\*k\*w\*t));  ezplot(xx,[0 10])  title('approximation with 11 terms') | lab910.bmp |
| k=-10:10;  a=(1/T)\*int(x\*exp(-j\*k\*w\*t),t,t0,t0+T);  xx=sum(a.\*exp(j\*k\*w\*t));  ezplot(xx,[0 10])  title('approximation with 21 terms') | lab911.bmp |
| k=-30:30;  a=(1/T)\*int(x\*exp(-j\*k\*w\*t),t,t0,t0+T);  xx=sum(a.\*exp(j\*k\*w\*t));  ezplot(xx,[0 10])  title('approximation with 61 terms') | lab912.bmp |

**In-Lab Tasks**

**Task 01: The periodic signal is defined in one period as . Plot in time of four periods the approximate signals using 81 terms of complex exponential form of Fourier series.**

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| t0 = 0;  T = 6;  w = 2.\*pi./T;  syms t  x = t.\*exp(-t);  subplot(3,1,1)  ezplot(x, [t0 t0+T])  title('Signal for 1 Period')  t1 = t0:0.01:T;  xx = t1.\*exp(-t1);  xrepeat = repmat(xx,1,4);  tt = linspace(0,4.\*T,length(xrepeat));  subplot(3,1,2)  plot(tt,xrepeat)  title('Signal with 4 periods')  for k = -40:40  a(k+41) = (1/T).\*int(x.\*exp(-j\*k\*w\*t), t, t0, t0+t);  end  for k = -40:40  ex(k+41) = exp(j\*k\*w\*t);  end  xx1 = sum(a.\*ex);  subplot(3,1,3)  ezplot(xx1, [t0 t0+4\*T])  title('Aproximation with 81 Terms')  Graphical user interface, chart  Description automatically generated |

**Task 02: Plot the coefficients of the complex exponential Fourier series for the periodic signal that in one period is defined by .**

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| t0 = -3;  T = 6;  w = 2.\*pi./T;  syms t  x = exp(-t.^2);  subplot(3,1,1)  ezplot(x, [t0 t0+T])  syms t k n  x = exp(-t);  a = (1/T)\*int(x.\*exp(-j\*k\*w\*t), t, t0, t0+T);  k1 = -3:3;  ak = subs(a, k, k1);  subplot(3,1,2)  stem(k1,abs(ak));  legend('|a\_k|')  subplot(3,1,3)  stem(k1,angle(ak));  legend('\angle a\_k')  Diagram  Description automatically generated with low confidence |

**Task 03: The periodic signal in a period is given by**

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**Plot in one period the approximate signals using 41 and 201 term of the complex exponential Fourier series. Furthermore, each time plot the complex exponential coefficients**.

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| t0 = 0;  T = 2;  w = 2\*pi/T;  syms t n k  x=heaviside(t)-heaviside(t-1)+heaviside(t-2)-heaviside(t-3);  ezplot(x, [-1 3])  title('heaviside(t)-heaviside(t-1)');  figure();  x=heaviside(t)-heaviside(t-1);    for k=-20:20  a(k+21)=(1/T)\*int(x\*exp(-1i\*k\*w\*t),t,t0,t0+T);  end  for k=-20:20  ex(k+21)=exp(1i\*w\*k\*t);  end  f=sum(a.\*ex);  ezplot(f,[-1 3])  title('approximated signal with 41 terms')  figure();    for k=-100:100  a(k+101)=(1/T)\*int(x\*exp(-1i\*k\*w\*t),t,t0,t0+T);  end  for k=-100:100  ex(k+101)=exp(1i\*w\*k\*t);  end  f2=sum(a.\*ex);  ezplot(f2,[-1 3])  title('approximated signal with 101 terms')  figure();    new\_int = -100:100;  ak=subs(a,k,new\_int);  subplot(2,1,1);  stem(new\_int,abs(a));  legend('|a\_k|')  subplot(2,1,2);  stem(new\_int,angle(a));  legend('\angle a\_k')  Chart  Description automatically generated  Graphical user interface, chart  Description automatically generated with medium confidence  Graphical user interface, chart  Description automatically generated  Graphical user interface  Description automatically generated |

**Task 04: The periodic signal in a period is given by**

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**Calculate the approximation percentage when the signal is approximated by 3, 5, 7, and 17 terms of the complex exponential Fourier series. Furthermore, plot the signal in each case.**

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| syms t k  x=(heaviside(t)-heaviside(t-1))+(2-t)\*(heaviside(t-1)-heaviside(t-2));  inter = -1:3;  ezplot(x, inter)  t0=0;  T=2;  w=2\*pi/T;    for k=-8:8  a(k+9)=(1/T)\*int(x\*exp(-1i\*k\*w\*t),t,t0,t0+T);  end  for k=-8:8  ex(k+9)=exp(1i\*w\*k\*t);  end  x2=sum(a.\*ex);  ezplot(x2,[-1 3])  title('approximated signal with 17 terms')    syms k  k1 = -8:8;  figure();  subplot(2,1,1);  stem(k1,abs(a));  legend('|a\_k|')  subplot(2,1,2);  stem(k1,angle(a));  legend('\angle a\_k')  Chart, line chart  Description automatically generated  Graphical user interface, chart, line chart  Description automatically generated  Chart  Description automatically generated |

**Post-Lab Task**

## Critical Analysis / Conclusion

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| In this lab, we learnt the effect of increased number of terms on graphs of signals, we also found coefficients “ak” complex Fourier series in MATLAB. We plotted and observed the complex exponential Fourier series. Moreover, we observed signals approximated in more than one terms of Fourier Series. |

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| **Lab Assessment** | | |
| **Pre-Lab** | **/1** | **/10** |
| **In-Lab** | **/5** |
| **Critical Analysis** | **/4** |
| **Instructor Signature and Comments** | | |